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Asymptotic Behavior for Two-Dimensional, Quasi-autonomous, Almost-Periodic Evolution Equations

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Let A be a maximal monotone operator in R^2 and $f(t)$: $R \to R^2$ any S^1-almost-periodic function. If u is a solution of the evolution equation $du/dt + Au(t) \ni f(t)$ which is bounded on R^+, then $u(t)$: $R^+ \to R^2$ is asymptotically almost periodic as $t \to +\infty$. © 1987 Academic Press, Inc.

INTRODUCTION

Let A be a maximal monotone operator in the two-dimensional euclidean space R^2 and $f = R \to R^2$ a measurable function which is S^1-almost periodic. We consider the quasi-autonomous evolution equation

$$\frac{du}{dt} + Au(t) \ni f(t). \tag{1}$$

The following facts are well-known:

— Any weak solution of (1) on some interval $J \subset R$ is, in fact, a strong, absolutely continuous solution (cf. [3, Proposition 3.8, p. III-30]).

— Either all solutions of (1) are unbounded as $t \to +\infty$, or they are all bounded on R^+ and for any solution u of (1) on R^+, there exists an almost-periodic solution with range contained in the set $\overline{\text{conv}}\,(u(R^+))$ (cf. [7, Theorem 1, p. 296] or [10, Theorem 2]).

In [7], we asked (Problem 1, p. 299) whether a solution of (1) defined and bounded on R is automatically almost periodic. In this paper, we show that the answer is positive in the two-dimensional case.

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1. Notation and Preliminary Results

In the sequel, we shall denote by X the space of S^1-almost-periodic functions $R \to R^2$. For any $f \in X$, the hull $H(f)$ is defined by

$$H(f) = \overline{\bigcup_{\alpha \in R} \{T_\alpha(f)\}}^{X},$$

where $T_\alpha(f)(t) = f(t+\alpha)$, $\forall (t, \alpha) \in R \times R$.

For any $M \geqslant 0$, we define

$$K_M = \{u \in C(R, R^2),\ u \text{ is a solution of (1) on } R,\ \operatorname*{Sup}_{t \in R} \|u(t)\| \leqslant M\}.$$

The following result, which is implicit in [4] and [10] (in a more general framework), will be useful in the sequel.

Lemma 1.1 *The set*

$$\tilde{K}_M = \bigcup_{\alpha \in R} T_\alpha(K_M)$$

is sequentially precompact for the topology of uniform convergence on bounded intervals.

Proof. Let $v_n = T_{\alpha_n} u_n$ be any sequence in $\tilde{K}_M$, with $u_n \in K_M$ and $\alpha_n \in R$, $\forall n \in N$. As a consequence of diagonal procedure, we can replace $\{v_n\}$ by a subsequence such that $v_n(-m) \to w_m \in \overline{D(A)}$ for all $m \in N$ and $T_{\alpha_n} f \to g$ in X as $n \to +\infty$. Then it follows from the general stability properties of weak solutions (cf., e.g., [7, pp. 75–77 and especially inequality (21), p. 76]) that $v_n(t)$ converges uniformly on bounded intervals to a (weak) solution v of $dv/dt + Av(t) \ni g(t)$, $t \in R$.

Remark 1.2. (a) This result implies in particular that any trajectory $u \in K_M$ is *uniformly continuous*: $R \to R^2$. Similarly, we have that any solution u of (1) on R^+ which is *bounded* is in fact uniformly continuous: $R^+ \to R^2$.

(b) If u and v are two solutions of (1), the function $\|u(t) - v(t)\|$ is non-increasing, therefore tends to a limit as $t \to +\infty$. Similarly, if u and v are two solutions of (1) bounded on R, $\|u(t) - v(t)\|$ tends to a limit as $t \to -\infty$. These two properties will be used systematically without further reference in Section 2 below.

(c) The following result, which is valid in a more general framework, will reveal essential in the proof of the main result.

LEMMA 1.3. *Let $u_1, u_2, u_3 \in D(A)$ and $f_1 \in Au_1, f_2 \in Au_2, f_3 \in Au_3$ be such that u_1, u_2, u_3 are affinely independent and assume that we have*

$$\forall (i, j), \qquad \langle f_i - f_j, u_i - u_j \rangle = 0.$$

Then, there exists $a \in R^2$ and a skew-symmetric $L \in \mathscr{L}(R^2)$ such that $\forall u \in \mathrm{Int}(\mathrm{conv}(u_1, u_2, u_3))$, $Au = \{a + Lu\}$.

Outline of proof. Since $\mathrm{Int}(\mathrm{conv}(u_1, u_2, u_3)) \subset D(A)$, it is sufficient to check that for any u of the form $\sum_{i=1}^{3} t_i u_i$ with $0 < t_i < 1$, $\forall i \in \{1, 2, 3\}$ and $\sum_{i=1}^{3} t_i = 1$, we have

$$Au \subset \left\{ \sum_{i=1}^{3} t_i f_i \right\}.$$

Now for any $f \in Au$ and $j \in \{1, 2, 3\}$ we have

$$\left\langle f - \sum_{i=1}^{3} t_i f_i, u - u_j \right\rangle = \langle f - f_j, u - u_j \rangle \geqslant 0.$$

Since $0 \in \mathrm{Int}(\mathrm{conv}(u_1 - u,\ u_2 - u, u_3 - u))$ we conclude that $f = \sum_{i=1}^{3} t_i f_i$. Hence the proof of Lemma 1.3 is completed.

The proof of our main result will rely essentially on the following properties that we recall without proof.

THEOREM 1.4 (S. Bochner [2]). *A function $u \in C(R, E)$ with E any Banach space is almost periodic (in the usual sense) if and only if for any pair of sequences $\{s_n\}$, $\{\sigma_n\}$ in R there are common subsequences $\{s_k\}$, $\{\sigma_k\}$ such that $T_{s_k} u$ converges pointwise to some function v, and $T_{\sigma_k} v$ and $T_{s_k + \sigma_k} u$ converge pointwise to a same limit w.*

THEOREM 1.5 (A. M. Fink, cf. [5, Lemma 6.7, p. 104]). *Let $z: R \to C$ be continuous and such that $|z(t)| = 1$, $\forall t \in R$. Then $z: R \to R^2$ is almost periodic, if and only if there exists $a \in R$ and $\theta(t)$ almost periodic: $R \to R$ such that $z(t) = \exp(iat + i\theta(t))$.*

2. THE RESULT

Let A and f be as in the Introduction. We have the following result.

THEOREM 2.1. *If $u \in C(R, R^2)$ is a bounded solution of (1), then $u(t)$ is almost periodic: $R \to R^2$.*

The proof of Theorem 2.1 relies on the following

LEMMA 2.2. *Let $\omega(t)$, $\omega_1(t)$ and $\omega_2(t)$ be three solutions of* (1) *such that we have*

$$\|\omega_1(t)-\omega_2(t)\| = \rho > 0, \qquad \forall t \in R, \tag{2}$$

$$\|\omega(t)-\omega_i(t)\| = r_i > 0, \qquad \forall t \in R \quad \textit{for } i \in \{1, 2\}, \tag{3}$$

$$|r_1 - r_2| < \rho < r_1 + r_2, \tag{4}$$

$$\omega(t) \textit{ is almost periodic.} \tag{5}$$

Then any bounded solution on R of $dv/dt + Av(t) \ni g(t)$ is almost periodic for all $g \in H(f)$.

Proof of Lemma 2.2. Let $\Omega(t) = \operatorname{Int}\operatorname{conv}(\omega(t), \omega_1(t), \omega_2(t))$ for all $t \in R$. By Lemma 1.3, for almost all t there exists $a(t) \in R^2$ and $L(t) \in \mathscr{L}(R^2)$ with $(L(t))^* = -L(t)$ and

$$A|_{\Omega(t)} = a(t) + L(t)|_{\Omega(t)}.$$

Clearly, for θ close enough to t, $\Omega_\theta \cap \Omega_t$ is non-empty. It follows that $a(t)$ and $L(t)$, being locally constant, are in fact constant and we have

$$A|_{\Omega(t)} = a + L|_{\Omega(t)}, \qquad \forall t \in R \text{ for some } a \text{ and } L \text{ fixed.}$$

Now, let $y(t) = \sum_{j=0}^{2} t_j \omega_j(t)$, where $\omega_0 := \omega$ and $t_j > 0$ for $j \in \{0, 1, 2\}$ are such that $\sum_{j=0}^{2} t_j = 1$.

Then $y(t)$ is obviously a solution of the *affine* evolution equation

$$\frac{dy}{dt} = -Ly(t) - a + f(t).$$

Since $y(t)$ is bounded, it follows that y is almost periodic (cf., e.g. [5]). By letting t_0 and $t_2 \to 0$ in the formula defining $y(t)$, we obtain that $\omega_1(t)$ is almost periodic, and the same result is true for $\omega_2(t)$.

Finally, let $g \in H(f)$ and set $g = \lim_{n \to \infty} T_{\alpha_n} f$ in X. We may assume (if necessary by refining $\{\alpha_n\}$) that $T_{\alpha_n}\omega_j \to v_j$ in $C_B(R, R^2)$ with v_j a solution of the equation

$$\frac{dv}{dt} + Av(t) \ni g(t) \tag{6}$$

for all $j \in \{0, 1, 2\}$ and v_j almost periodic, with

$$\|v_1(t) - v_2(t)\| \equiv \rho > 0, \qquad \|v_0(t) - v_j(t)\| \equiv r_i > 0.$$

Let v be any solution of (6) bounded on R. Pick any sequence $\tau_n \to +\infty$ such that $g(t-\tau_n) \to g$ in X, $v_j(t-\tau_n) \to v_j(t)$ uniformly on R for $j \in \{0, 1, 2\}$ and $v(t-\tau_n) \to w(t)$ uniformly on bounded intervals of R (this is possible by Lemma 1.1). Clearly $\|w(t)-v_j(t)\|$ is constant on R for $j \in \{0, 1, 2\}$ and since we have (4), we can at least choose k and l such that $(w(t), v_k(t), v_l(t))$ form a proper triangle.

By applying the first part of the proof we obtain that $w(t)$ is an almost-periodic solution of (6).

We can assume that $w(t+\tau_r)$ converges uniformly as $r \to \infty$ to some almost-periodic solution $\tilde{v}(t)$ of (6).

Now we have the following properties:

(1) $\|v(t)-\tilde{v}(t)\|$ is non-increasing on R,

(2) $\lim_{r \to +\infty} \|\tilde{v}(-\tau_r)-v(-\tau_r)\| = 0$,

because $\tilde{v}(-\tau_r)$ and $v(-\tau_r)$ both tend to $w(0)$ as $t \to +\infty$.

Hence $v = \tilde{v}$, and the proof of Lemma 2.2 is completed.

Proof of Theorem 2.1. We will proceed in several steps.

Step 1. Let ω be an almost-periodic solution of (1) (which exists as recalled in the Introduction) and $v = \omega + z$ another solution of (1) on R such that

$$\forall t \in R, \qquad \|z(t)\| \equiv r > 0, \tag{7}$$

$$z(t) \text{ is } \textit{not} \text{ almost periodic.} \tag{8}$$

Then $\omega(t) - z(t)$ is a solution of (1).

Proof. Equation (8) implies the existence of a sequence $\{\alpha_n\}$ of reals such that the sequence $T_{\alpha_n} z$ has no Cauchy subsequence in $C_B(R, R^2)$. We can assume, by a suitable refining of $\{\alpha_n\}$, that $f(t+\alpha_n) \to f'(t)$ in X and $\omega(t+\alpha_n) \to \omega'(t)$ in $C_B(R, R^2)$, while $z(t+\alpha_n) \to z'(t)$ uniformly on bounded intervals of R. Since the sequence $T_{\alpha_n} z$ is *not* a Cauchy sequence in $C_B(R, R^2)$, there exists $\varepsilon > 0$, a sequence $\{t_n\}$ of reals and two subsequences $\{\tau_n\}$ and $\{\theta_n\}$ of $\{\alpha_n\}$ such that

$$\|z(\tau_n + t_n) - z(\theta_n + t_n)\| \geqslant \varepsilon, \tag{9}$$

$$z(\tau_n + t_n + t) \to z_1(t) \qquad \text{uniformly on bounded intervals,} \tag{10}$$

$$z(\theta_n + t_n + t) \to z_2(t) \qquad \text{uniformly on bounded intervals,} \tag{11}$$

$$f'(t_n + t) \to f''(t) \qquad \text{in } X, \tag{12}$$

$$\omega'(t_n + t) \to \omega''(t) \qquad \text{in } C_B(R, R^2). \tag{13}$$

It is obvious that ω'', $\omega''+z_1$ and $\omega''+z_2$ are three different solutions of

$$\frac{du}{dt}+Au(t)\ni f''(t). \tag{14}$$

We now choose a sequence $\beta_n\to+\infty$ such that

$$f''(t-\beta_n)\to f''(t) \qquad \text{in } S^1(R, R^2)=X, \tag{15}$$

$$\omega''(t-\beta_n)\to\omega''(t) \qquad \text{in } C_B(R, R^2) \tag{16}$$

[this is possible since the vector (f'', ω'') is S^1 almost periodic: $R\to R^2\times R^2$], and in addition

$$z_i(t-\beta_n)\to y_i(t) \qquad \text{uniformly on bounded intervals for } i\in\{1,2\}. \tag{17}$$

Then ω'', $\omega''+y_1$ and $\omega''+y_2$ are three different bounded solutions of (14) which remain at a constant distance from each other.

Finally:

— either the triangle formed by these three solutions is not "flat," and by Lemma 2.2, all solutions of (1) are almost periodic; this contradicts the hypothesis;

— or this triangle is "flat," and since $y_1\neq y_2$ and $\|y_1\|=\|y_2\|$ the only possibility is that we have $y_2=-y_1$.

From y_1 we deduce $y\in C_B(R, R^2)$ such that $\omega+y$ and $\omega-y$ are both solutions of (1), by using the translation $T_{\beta_n-\alpha_n-t_n}$ which carries f'' and ω'' back to f and ω, respectively.

Then by going along a negative translation $T_{-\tau_n}$ such that $T_{-\tau_n}f\to f$ and $T_{-\tau_n}\omega\to\omega$ in X and $C_B(R, R^2)$, respectively, we obtain $\tilde{z}$ such that $\omega+\tilde{z}$ and $\omega-\tilde{z}$ are solutions of (1) with $\|z-\tilde{z}\|$ and $\|z+\tilde{z}\|$ constant and $\|\tilde{z}(t)\|\equiv\|z(t)\|\equiv r$.

This concludes the proof since the only way for the triangles $(0, z, \tilde{z})$ and $(0, z, -\tilde{z})$ to be flat is that $\tilde{z}=\pm z$.

If these triangles are *not* flat, we have again a contradiction.

Step 2. Under the hypotheses of Step 1, for any $(g, \gamma)\in H(f, \omega)$, there exists $z_{g,\gamma}\in C(R, R^2)$ such that $\|z_{g,\gamma}(t)\|\equiv r$ on R and $\gamma\pm z_{g,\gamma}$ are solutions of (6).

Proof. Obvious from the previous result.

Step 3. Under the hypotheses of Step 1, for any $(g, \gamma) \in H(f, \omega)$ the conditions

$$y \in C(R, R^2), \quad \|y(t)\| \equiv r \qquad \text{on } R, \tag{18}$$

$$\gamma + y \text{ is a solution of (6) on } R \tag{19}$$

$$\text{imply: } y = \pm z_{g,\gamma}.$$

Proof. Since $\gamma \pm z_{g,\gamma}$ *are two solutions of* (6), the norms $\|y(t) - z_{g,\gamma}(t)\|$ and $\|y(t) + z_{g,\gamma}(t)\|$, which are both non-increasing, must in fact be *constant.* Hence if $y \neq \pm z_{g,\gamma}$, we find again a non-trivial triangle of solutions of (6) remaining at a constant distance from each other, and Lemma 2.2 gives a contradiction.

Step 4. Let ω be an almost-periodic solutions of (1) and $v = \omega + z$ a solution of (1) on R such that $\|z(t)\| \equiv r > 0$. Then v is almost periodic: $R \to R^2$.

Proof. Assuming that $z(t)$ is not almost periodic, the result of Step 3 together with Theorem 1.4 show that the complex-valued function $z^2(t)$ is almost periodic.

By Theorem 1.5, we obtain

$$z^2(t) = r^2 \exp(iat + i\theta(t))$$

for some $a \in R$, θ almost periodic $R \to R$.

Now $y(t) = z(t) \exp(-i(a/2)t - i(\theta(t)/2))$ is continuous and takes only the values $-r$ and $+r$. Hence it is constant on R, and $z(t)$ is almost periodic. This contradiction completes the proof of Step 4.

Step 5. Let u be as in the statement of Theorem 2.1 and pick $\tau_n \to +\infty$ such that $f(t - \tau_n) \to f$ in X, $\omega(t - \tau_n) \to \omega(t)$ in $C_B(R, R^2)$ and $u(t - \tau_n) \to v(t)$ uniformly on compact intervals.

Then v is a solution of (1) and $z = v - \omega$ has constant norm. It follows that $v(t)$ is almost periodic. Let $v(t + \sigma_k) \to \tilde{u}(t)$ in $C_B(R, R^2)$ for some subsequence $\{\sigma_k\}$ of $\{\tau_n\}$. Then it is easy to check that $u = \tilde{u}$. Hence u is almost periodic and the proof of Theorem 2.1 is completed.

COROLLARY 2.3. *Any solution of* (1) *which is bounded*: $R^+ \to R^2$ *is asymptotic to an almost-periodic solution of* (1) *as* $t \to +\infty$.

Proof. Standard argument, consisting in two successive "infinite translations," the first one towards $(+\infty)$ the second one "back" towards $(-\infty)$ along the opposite sequence.

3. An Example

Let β be a maximal monotone graph in $R \times R$ and f: $R \to R$ be S^1-almost periodic, k a positive real number. Consider the second order ODE:

$$u'' + k^2 u + \beta(u') \ni f(t). \tag{20}$$

This equation is equivalent to the system

$$\begin{aligned} &u' - kv = 0, \\ &v' + ku + \frac{1}{k}\beta(kv) \ni \frac{1}{k} f(t), \end{aligned} \tag{21}$$

which can be written as an evolution equation in R^2

$$\frac{dU}{dt} + AU(t) \ni F(t) \tag{22}$$

with $U(t) = (u(t), v(t))$, $D(A) = D(\beta) \times R$ and

$$A(u, v) = \left(-kv, ku + \frac{1}{k}\beta(kv)\right), \qquad \forall (u, v) \in D(A),$$

$$F(t) = \left(0, \frac{1}{k} f(t)\right), \qquad \forall t \in R.$$

It is readily verified that A is maximal monotone in R^2 and $F(t) \in X$. As a consequence of Corollary 2.3, any solution $U(t)$ of (22) which is bounded for $t \geqslant 0$ is asymptotically almost periodic as $t \to +\infty$.

We conclude that any solution $u \in W^{1,\infty}(R^+)$ of Eq. (20) is such that $u(t)$ and $u'(t)$ are asymptotically almost periodic as $t \to +\infty$.

Remark 3.1. (a) This example generalizes [6, pp. 224–225] to the almost-periodic case.

(b) It would be interesting to decide whether an analogous property holds true for the partial differential equation

$$\begin{aligned} &u_{tt} - \Delta u + \beta(u_t) \ni f(t, x), \\ &u|_{\partial\Omega} = 0. \end{aligned} \tag{23}$$

Partial results in this direction can be found in [6] and [9].

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